E2.5 Signals & Linear Systems

Tutorial Sheet 3 - Solutions

1. (a) $u(t) * u(t)$

$$= \int_0^t u(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_0^t d\tau = \tau \Big|_0^t = t \qquad \text{for } t \geq 0$$

$$= 0 \qquad \text{for } t < 0$$

$\therefore\ u(t) * u(t) = t\, u(t)$

(b) $e^{-at} u(t) * e^{-bt} u(t)$

$$= \int_0^t e^{-a\tau} e^{-b(t-\tau)} d\tau = e^{-bt} \int_0^t e^{(b-a)\tau} d\tau$$

$$= \frac{e^{-bt}}{b-a} e^{(b-a)\tau} \Big|_0^t = \frac{e^{-bt}}{b-a} \left[e^{(b-a)t} - 1 \right]$$

$$= \frac{e^{-at} - e^{-bt}}{a-b}$$

Because both functions are casual, their convolution is zero for $t < 0$.

$\therefore\ y(t) = e^{-at} u(t) * e^{-b(t)} u(t)$

$$= \left(\frac{e^{-at} - e^{-bt}}{a-b} \right) u(t).$$

(c) Both functions are causal, $\therefore$

$$t\,u(t) * u(t) = \int_0^t \tau\, u(\tau)\, u(t-\tau)\, d\tau$$

For the range of integration, $0 \le \tau \le t$,

and $u(\tau)$ at $\tau = 0$ is $u(0) = 1$, at $\tau = t$ $u(t) = 1$

and $u(t-\tau)$ at $\tau = 0$ is $u(t) = 1$, at $\tau = t$, $u(0) = 1$

$\therefore$ $u(\tau) = u(t-\tau) = 1$, and

$$\cancel{t\,u} \; t\,u(t) * u(t) = \int_0^t \tau\, d\tau$$

$$= \frac{t^2}{2} \quad \text{for } t \geq 0$$

$$= 0 \quad \text{for } t < 0$$

$\therefore$ $\underline{y(t) = \frac{1}{2} t^2 u(t)}$.

2. a) $y(t) = \sin t\, u(t) * u(t)$

$$= \left[\int_0^t \sin\tau\, u(\tau)\, u(t-\tau)\, d\tau\right] u(t)$$

$$= \left[\int_0^t \sin\tau\, d\tau\right] u(t) \qquad \begin{array}{l} u(\tau) = 1 \\ u(t-\tau) = 1 \end{array}$$

$$= (1 - \cos t)\; u(t).$$

b) Similarily,

$$y(t) = \left[\int_0^t \cos\tau\, d\tau\right] u(t) = \sin t\; u(t)$$

3. a) $h(t) = e^{-t}u(t)$, $f(t) = u(t)$

$$y(t) = e^{-t}u(t) * u(t) \qquad \text{(Pair \#2)}$$

$$= (1 - e^{-t})\,u(t)$$

b) $f(t) = e^{-2t}u(t)$

$$y(t) = e^{-2t}u(t) * e^{-t}u(t) \qquad \text{(Pair \#4)}$$

$$= (e^{-t} - e^{-2t})\,u(t)$$

c) $f(t) = \sin 3t \; u(t)$

$$y(t) = \sin 3t \; u(t) * u^{-t}u(t)$$

(Pair #12)

$\alpha = 0,\ \beta = 3,\ \theta = -90^\circ,\ \lambda = -1.$

$$\therefore\ \phi = \tan^{-1}\left[\frac{-3}{-1}\right] = 108.4^\circ$$

and
$$y(t) = \frac{\cos(-90^\circ + 108.4^\circ)e^{-t} - \cos(3t + 18.4^\circ)}{\sqrt{(0-1)^2 + 3^2}}$$

$$= \frac{0.9485e^{-t} - \cos(3t - 18.4^\circ)}{\sqrt{10}}\,u(t)$$

— · — · — Pair #12 — · — · —

$$e^{-\alpha t}\cos(\beta t + \theta)u(t) * e^{\lambda t}u(t)$$

$$\Longrightarrow \frac{\cos(\theta - \phi)e^{\lambda t} - e^{-\alpha t}\cos(\beta t + \theta - \phi)}{\sqrt{(\alpha + \lambda)^2 + \beta^2}}\,u(t)$$

$$\phi = \tan^{-1}\left[-\beta/(\alpha + \lambda)\right]$$

4. The key is to realise that

$$f(t) = u(t) - u(t-1).$$

$$\therefore \; e^{-t}u(t) * u(t) \Longleftrightarrow \underbrace{(1-e^{-t})u(t)}_{z(t)} \quad \text{A}$$

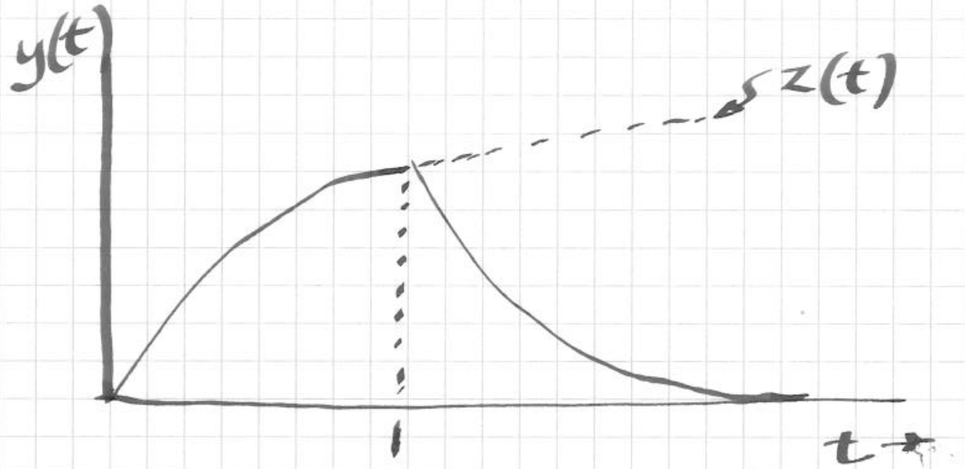
$$e^{-t}u(t) * u(t-1) \Longrightarrow z(t-1) \quad \text{B}$$

$$= \left[1-e^{-(t-1)}\right]u(t-1)$$

$$\therefore \; y(t) = e^{-t}u(t) * \left[u(t) - u(t-1)\right]$$

$$= (A - B) \qquad \text{superposition}$$

$$= (1-e^{-t})u(t) - \left[1-e^{-(t-1)}\right]u(t-1)$$



Q5a) $\cdot\ h(t) = -\delta(t) + 2e^{-t}u(t)$

$x(t) = e^{t}u(-t)$

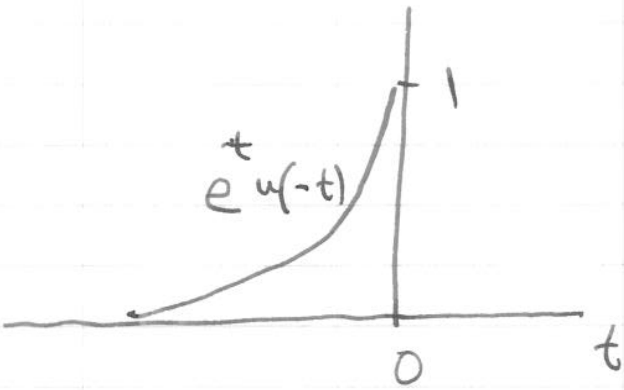
$$y(t) = h(t) * x(t)$$
$$= \left[-\delta(t) + 2e^{-t}u(t)\right] * e^{t}u(-t)$$
$$= -\delta(t) * e^{t}u(-t) + 2e^{-t}u(t) * e^{t}u(-t)$$

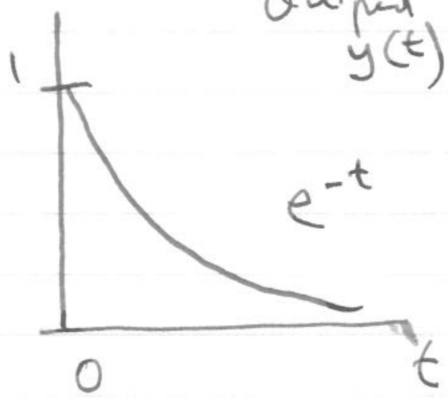
See convolution table no: 13, lecture 5-5

$$-e^{t}u(-t) \qquad\qquad 2\left[\frac{e^{-t}u(t) + e^{t}u(-t)}{2}\right]$$

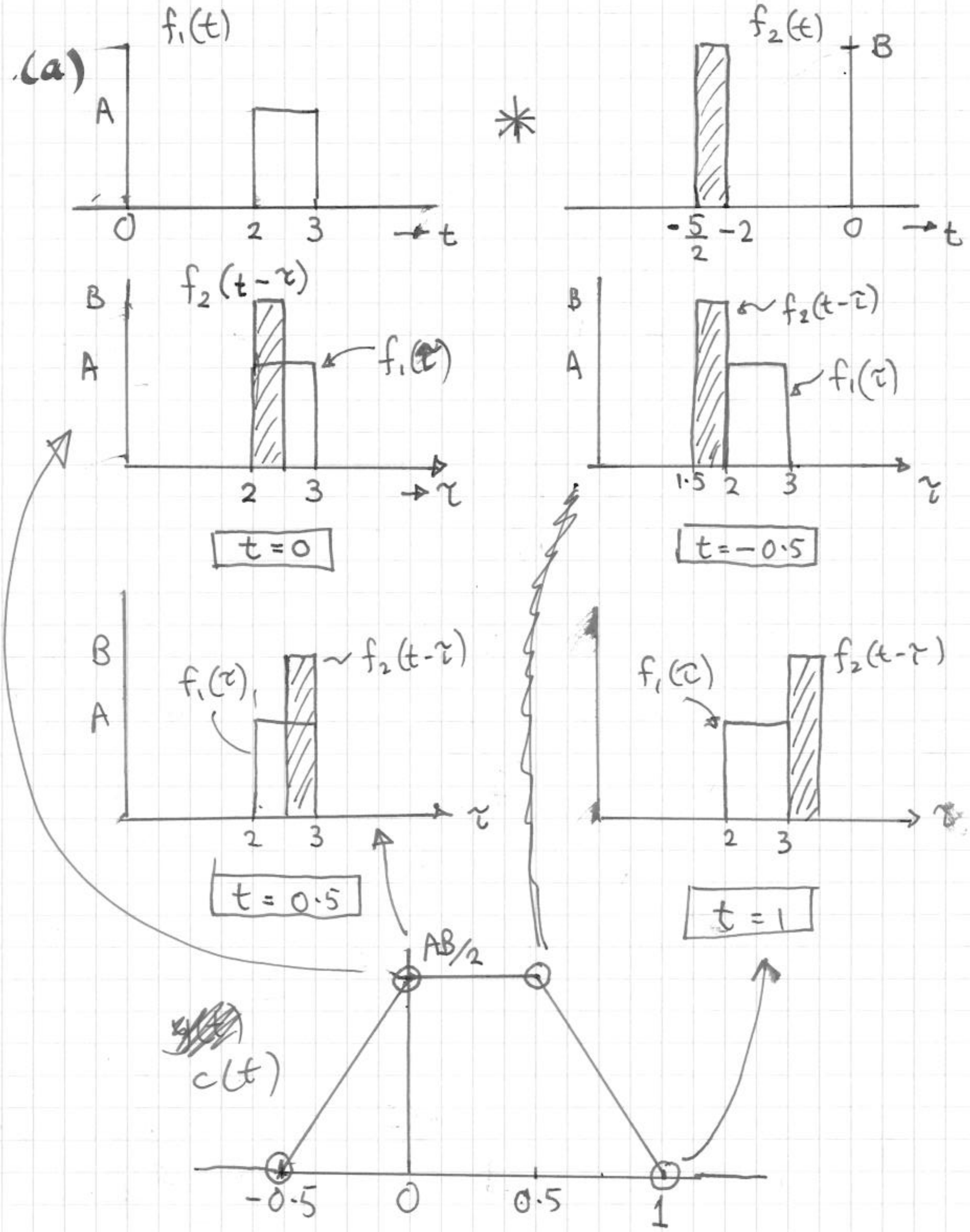
$$\therefore y(t) = \cancel{-e^{t}u(-t)} + e^{-t}u(t) + \cancel{e^{t}u(-t)}$$
$$= e^{-t}u(t)$$

b) input $x(t)$



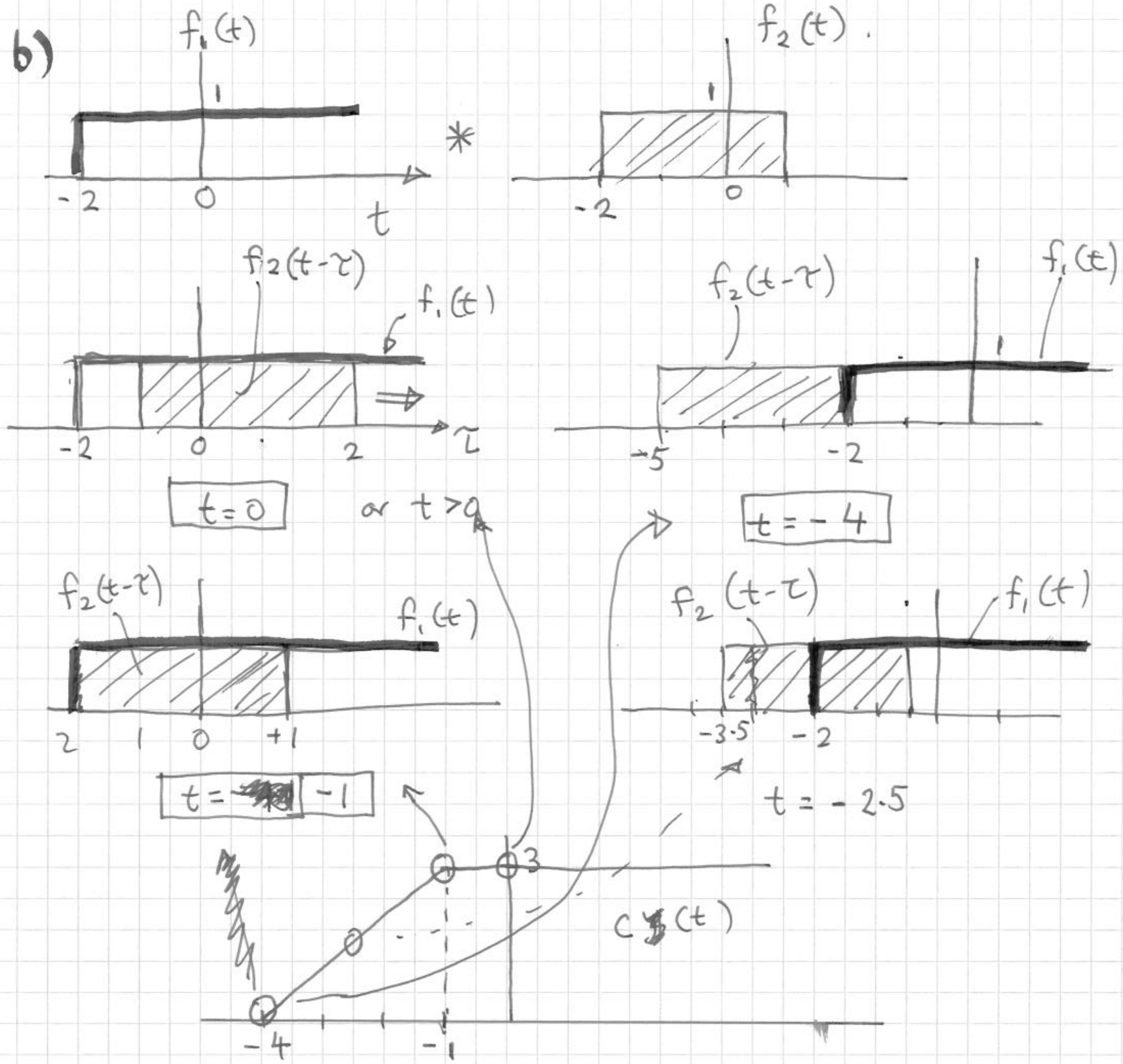


Q6 .(a)



$f_1(t)$
A
0
2
3
t
*
$f_2(t)$
B
$-\frac{5}{2}$
-2
0
t
$f_2(t-\tau)$
B
A
$f_1(\tau)$
2
3
τ
t = 0
B
A
$f_2(t-\tau)$
$f_1(\tau)$
1.5
2
3
τ
t = −0.5
B
A
$f_1(\tau)$
$f_2(t-\tau)$
2
3
τ
t = 0.5
$f_1(\tau)$
$f_2(t-\tau)$
2
3
t = 1
AB/2
c(t)
-0.5
0
0.5
1

6. b)

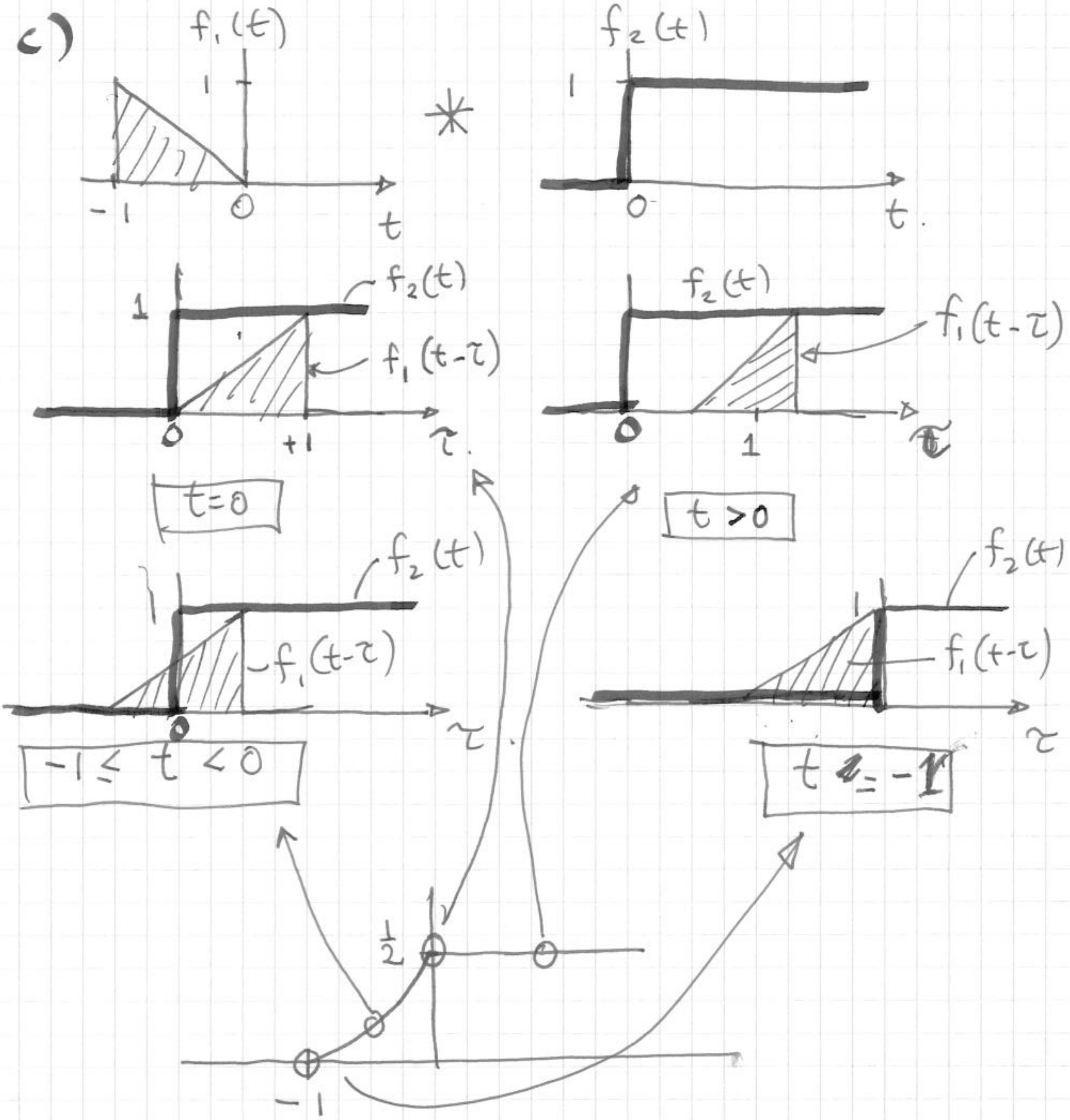


$$c(t) = \int_{-1+t}^{2+t} d\tau = 3 \qquad t \geq -1$$

$$c(t) = \int_{-2}^{2+t} d\tau = t+4 \qquad -1 \geq t \geq -4$$

$$c(t) = 0 \qquad t \leq -4$$

6 c)



Note : Easy to do $\int f_2(\tau)\, f_1(t-\tau)\, d\tau$

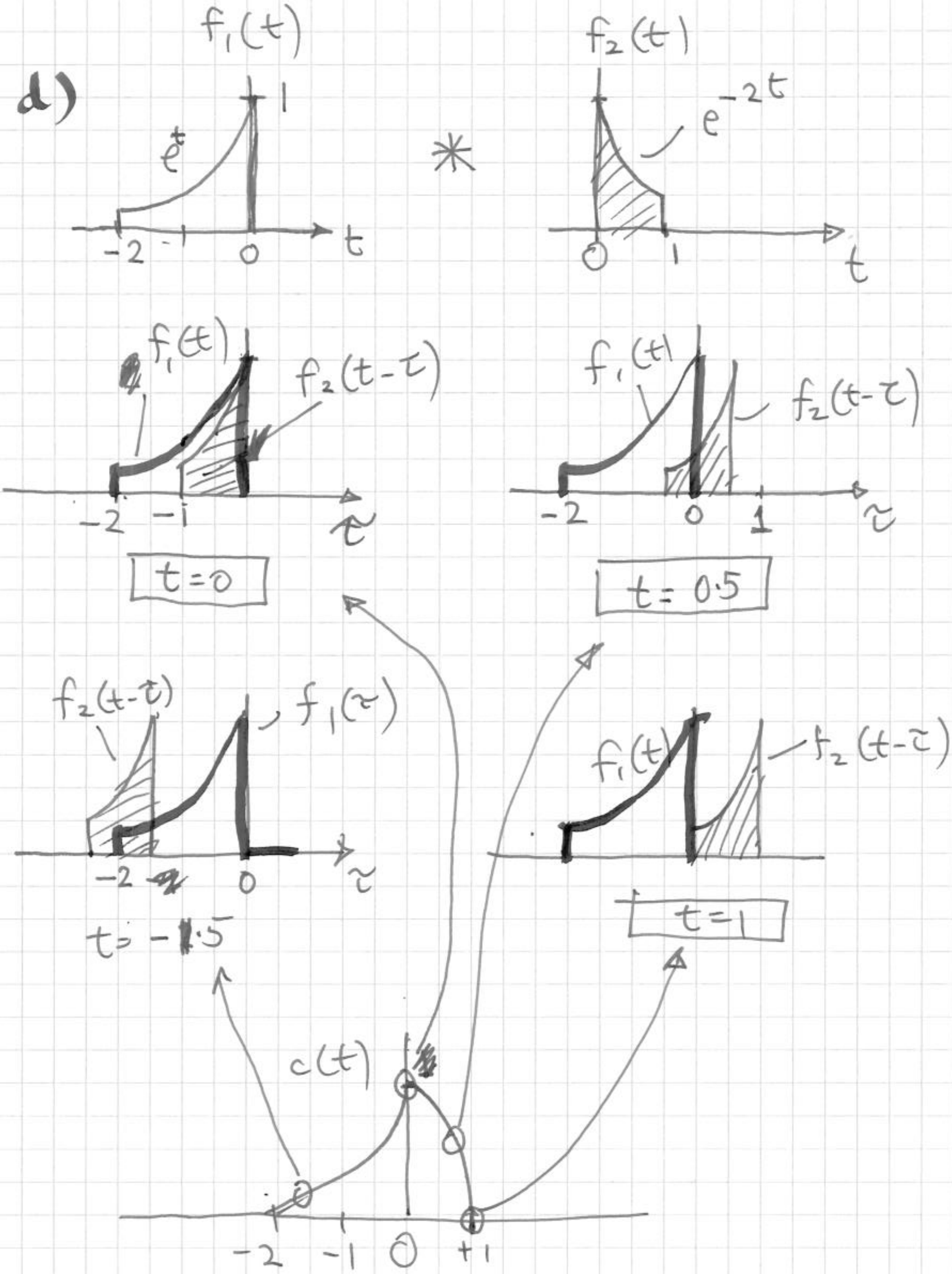
$$c(t) = \int_t^{t+1} (\tau - t)\, d\tau = \frac{1}{2} \qquad t \geq 0$$

$$c(t) = \int_0^{t+1} (\tau - t)\, d\tau = \frac{1}{2}(1 - t^2)$$

$$0 \geq t \geq -1$$

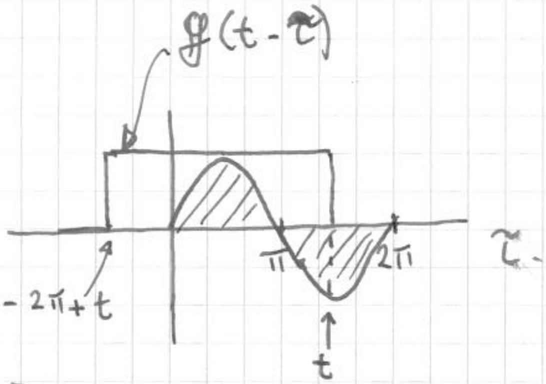
$$c(t) = 0 \qquad t < -1$$

6 d)



S3.9

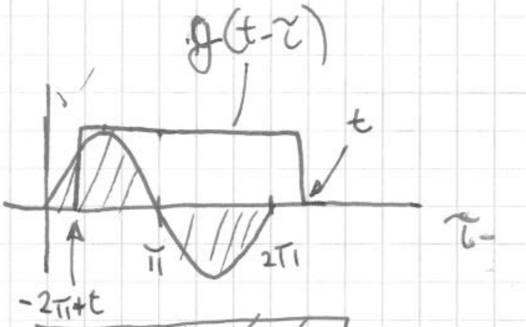
7)



$\boxed{0 \le t \le 2\pi}$

$$f(t)*g(t) = \int_0^t \sin\tau \, d\tau \qquad 0 \le t \le 2\pi$$

$$= 1 - \cos t$$

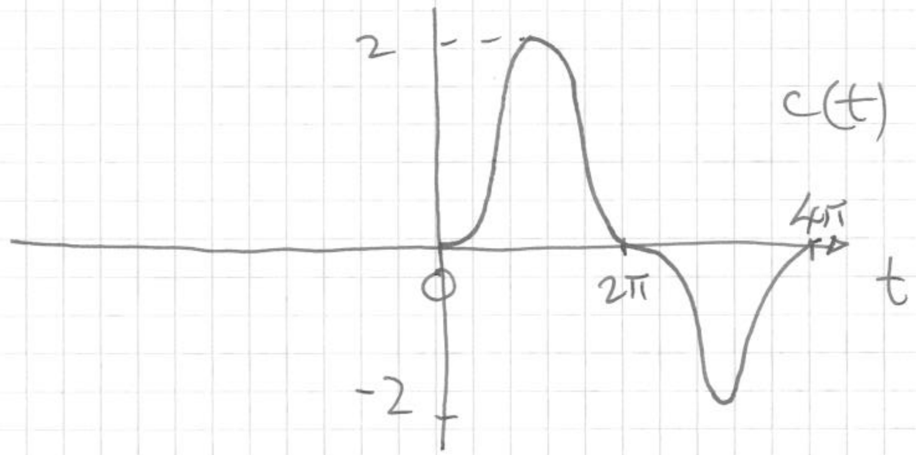


$\boxed{\cancel{4\pi \ge t > 2\pi}}$

$\boxed{2\pi < t \le 4\pi}$

$$f(t)*g(t) = \int_{t-2\pi}^{2\pi} \sin r \, dr \qquad 2\pi \le t \le 4\pi$$

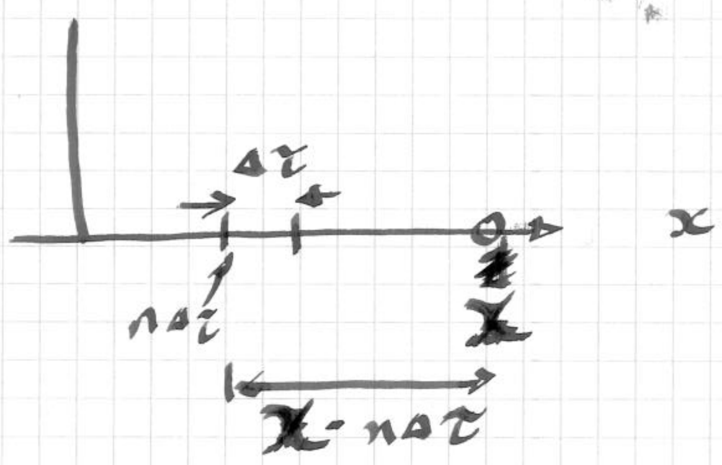
$$= \cos t - 1$$



53·10

~~53·9~~

8.



An element of length $\Delta\tau$ at point $n\Delta\tau$ has a charge $f(n\Delta\tau)\,\Delta\tau$.

The electric field due to this charge at point x is

$$\Delta E = \frac{f(n\Delta\tau)\Delta\tau \times q}{4\pi\varepsilon\,(x - n\Delta\tau)^2}$$

$\therefore$ Total Field E due to charge along the entire length is

$$E(x) = \lim_{\Delta\tau \to 0} \sum_{n=-\infty}^{\infty} \frac{f(n\Delta\tau)\Delta\tau \times q}{4\pi\varepsilon(x - n\Delta\tau)^2}$$

$$= \int_{-\infty}^{\infty} \frac{f(\tau)\; q}{4\pi\varepsilon(x-\tau)^2}\, d\tau$$

$$= f(x) * \frac{q}{4\pi\varepsilon x^2}$$

$$= f(x) * h(x)$$